1 Generalized Stirling Numbers of Type \((-1, -d, 0)\)

In Appendix A.5 of [1] it was shown by induction that
\[
\sum_{A \in \mathcal{A}_c} \prod_{a \in A} [1 - d_1^{a|a|-1} = S_d(c, t)
\]
where \(S_d(c, t)\) is a generalized Stirling number of type \((-1, -d, 0)\) [2]. These can be computed recursively as follows:
\[
S_d(1, 1) = S_d(0, 0) = 1
\]
\[
S_d(c, 0) = S_d(0, t) = 0
\]
\[
S_d(c, t) = 0
\]
\[
S_d(c, t) = S_d(c - 1, t - 1) + (c - 1 - dt)S_d(c - 1, t)
\]
for \(0 < t \leq c\)

2 More Verbose Proof of Theorem 1

**Theorem 1.** Suppose \(A_2 \in \mathcal{A}_c\), \(A_1 \in \mathcal{A}_{[A_2]}\), \(C \in \mathcal{A}_c\), and \(F_a \in \mathcal{A}_{|a|}\) for each \(a \in C\) are related as above. Then the following describe equivalent distributions: (I) \(A_2 \sim \text{CRP}_c(\alpha d_2, d_2)\) and \(A_1|A_2 \sim \text{CRP}_{[A_2]}(\alpha, d_1)\). (II) \(C \sim \text{CRP}_c(\alpha d_2, d_1d_2)\) and \(F_a|C \sim \text{CRP}_{[a]}(-d_1 d_2, d_2)\) for each \(a \in C\).

**Proof.** In the proof we will use the identity \([\beta + \delta]^{n-1} = \delta^{n-1}[\beta + 1]^{n-1}\) (for all \(\beta, \delta, n\)) several times; let us call it Identity 1.

To complete the proof, we simply show that the joint distributions are the same. Starting with the definition of the CRP distribution
\[
P(A) = \frac{[\alpha + d_1]^{A_1|A_1}|A_1|A_1-1}{[\alpha + 1]^{A_1|A_1-1}} \prod_{a \in A} [1 - d_1^{a|a|-1}
\]
for each \(A \in \mathcal{A}_c\),

we have the following by multiplying the two distributions together:
\[
P(A_1, A_2) = \left(\frac{[\alpha + d_1]^{A_1|A_1}|A_1|A_1-1}{[\alpha + 1]^{A_1|A_1-1}} \prod_{a \in A_1} [1 - d_1^{a|a|-1}\right) \left(\frac{[\alpha d_2 + d_2]^{A_2|A_2-1}}{[\alpha d_2 + 1]^{A_2|A_2-1}} \prod_{b \in A_2} [1 - d_2^{b|b|-1}\right)
\]
Re-arranging terms yields:
\[
= \frac{[\alpha + d_1]^{A_1|A_1}|A_1|A_1-1}{[\alpha + 1]^{A_1|A_1-1}} \frac{[\alpha d_2 + d_2]^{A_2|A_2-1}}{[\alpha d_2 + 1]^{A_2|A_2-1}} \left(\prod_{a \in A_1} [1 - d_1^{a|a|-1}\right) \left(\prod_{b \in A_2} [1 - d_2^{b|b|-1}\right)
\]
Using Identity 1 on $[\alpha d_2 + d_2]^{\alpha A_2}_{d_2}$ yields
\[
= \frac{[\alpha + d_1]^{\alpha A_1}_{d_1}}{[\alpha + 1]^{\alpha A_1}_{d_1}} \frac{d_2^{\alpha A_2}_{d_2}}{[\alpha d_2 + 1]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} [1 - d_1]^{\alpha A_1}_{d_1} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(9)

Cancelling one of the resulting terms we get:
\[
= \frac{\alpha + d_1}{\alpha + 1} \frac{\alpha d_2^{\alpha A_2}_{d_2}}{[\alpha d_2 + 1]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} [1 - d_1]^{\alpha A_1}_{d_1} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(10)

Multiplying and dividing by $d_2^{\alpha A_2}_{d_2}$ within the first product and using Identity 1 again we get:
\[
= \frac{\alpha + d_1}{\alpha + 1} \frac{d_2^{\alpha A_2}_{d_2}}{[\alpha d_2 + 1]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(11)

Using $\sum_{a \in A_1} |a| = |A_2|$ to take the $1/(d_2^{\alpha} - 1)$ term out of the first product:
\[
= \frac{\alpha + d_1}{\alpha + 1} \frac{d_2^{\alpha A_2}_{d_2}}{[\alpha d_2 + 1]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(12)

Using $d_2^{\alpha A_2}_{d_2} = d_2^{\alpha A_2}_{d_2} - d_2^{\alpha A_2}_{d_2}$ and cancelling the $d_2^{\alpha A_2}_{d_2}$ term:
\[
= \frac{\alpha + d_1}{\alpha + 1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(13)

Multiplying and dividing by $d_2^{\alpha A_2}_{d_2}$ and using Identity 1 a third time we get:
\[
= \frac{\alpha d_2 + d_1 d_2}{\alpha d_2 + 1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(14)

Finally, cancelling terms again we get:
\[
= \frac{\alpha d_2 + d_1 d_2}{\alpha d_2 + 1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \left( \prod_{a \in A_1} \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \left( \prod_{b \in A_2} [1 - d_2]^{\alpha A_2}_{d_2} \right)
\]
(15)

Re-grouping the products and expressing the same quantities in terms of $C$ and $\{F_a\}$,
\[
= \frac{\alpha d_2 + d_1 d_2}{\alpha d_2 + 1} \prod_{a \in C} \left( \frac{1}{d_2^{\alpha} [d_2 - d_1 d_2]^{\alpha A_2}_{d_2}} \right) \prod_{a \in F_a} [1 - d_2]^{\alpha A_2}_{d_2} = P(C; \{F_a\}_{a \in C})
\]
(16)

By comparison with (6), we see that conditioning on $C$ each $F_a \sim CRP_{a_{-}}(-d_1 d_2, d_2)$.

Marginalizing $\{F_a\}$ out using the normalization constant from (6) we have:
\[
P(C) = \frac{\alpha d_2 + d_1 d_2}{\alpha d_2 + 1} \prod_{a \in C} [1 - d_2]^{\alpha A_2}_{d_2}
\]
(17)

So $C \sim CRP_{\alpha_{-}}(\alpha d_2, d_1 d_2)$ and (I)$\Rightarrow$(II). Reversing the same argument shows that (II)$\Rightarrow$(I).
References
