We first prove equation (4) of the main text for a general nonstationary hazard function \( h(\tau, t) \).

**Proposition S.1** For a renewal process with nonstationary hazard function \( h(\tau, t) \), the waiting time \( \tau \) given that the last event occurred at time \( t_{\text{prev}} \) is given by

\[
g(\tau | t_{\text{prev}}) = h(\tau, t_{\text{prev}} + \tau) \exp \left( -\int_0^\tau h(u, t_{\text{prev}} + u) du \right)
\]  

(1)

**Proof.** By definition (see equation (2) in the main text),

\[
h(\tau, t_{\text{prev}} + \tau) = \frac{g(\tau | t_{\text{prev}})}{1 - \int_0^\tau g(u | t_{\text{prev}}) du}
\]  

(2)

Let \( y = 1 - \int_0^\tau g(u | t_{\text{prev}}) du \). It follows that

\[
h(\tau, t_{\text{prev}} + \tau) = \frac{-dy/d\tau}{y}, \text{ so that}
\]

\[
y = \exp \left( -\int_0^\tau h(u, t_{\text{prev}} + u) du \right)
\]  

(4)

Substituting back for \( y \) and differentiating w.r.t. \( \tau \), we get equation (1). \( \square \)

We now prove proposition 2 from the main text.

**Proposition 2** For any \( \Omega \geq \max_{t, \tau} h(\tau)\lambda(t) \), \( F \) is a sample from a modulated renewal process with hazard \( h(\cdot) \) and modulating intensity \( \lambda(\cdot) \).

**Proof.** We need to show that \( F_i - F_{i-1} \sim g \).
Denote by \( E_i^* \) the restriction of \( E \) to the interval \( (F_{i-1}, F_i) \), not including boundaries. Note that

\[
P(F_i, E_i^* | F_{i-1}) = \left( \prod_{e \in E_i^*} 1 - \frac{\lambda(e)h(e - F_{i-1})}{\Omega} \right) \frac{\lambda(F_i)h(F_i - F_{i-1})}{\Omega} \)  

(5)
Defining \( n = |E_i^*| \) and \( t_0 = F_{i-1} \), we have

\[
P(F_i, n|F_{i-1}) = \frac{\lambda(F_i)h(F_i - F_{i-1})}{\Omega}
\]

\[
\int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} \ldots \int_{t_{n-1}}^{F_i} dt_1 dt_2 \ldots dt_n \left( \prod_{j=1}^{n} \Omega \exp -\Omega(t_j - t_{j-1}) \right) \left( \prod_{j=1}^{n} \left( 1 - \frac{\lambda(t_j)h(t_j - F_{i-1})}{\Omega} \right) \right) (\Omega \exp -(\Omega(F_i - t_n)))
\]

\[
= \lambda(F_i)h(F_i - F_{i-1})\exp (-\Omega(F_i - F_{i-1})) \int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} \ldots \int_{t_n}^{F_i} dt_1 dt_2 \ldots dt_n \left( \prod_{j=1}^{n} (\Omega - \lambda(t_j)h(t_j - F_{i-1})) \right)
\]

\[
= \lambda(F_i)h(F_i - F_{i-1})\exp (-\Omega(F_i - F_{i-1})) \frac{1}{n!} \left( \int_{F_{i-1}}^{F_i} dt (\Omega - \lambda(t)h(t - F_{i-1})) \right)^n
\]

(6)

Marginalizing out \( n \), we then have

\[
P(F_i|F_{i-1}) = \lambda(t)h(F_i - F_{i-1})\exp (-\Omega(F_i - F_{i-1})) \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{F_{i-1}}^{F_i} dt (\Omega - \lambda(t)h(t - F_{i-1})) \right)^n \right)
\]

\[
= \lambda(F_i)h(F_i - F_{i-1}) \exp \left( -\int_{F_{i-1}}^{F_i} \lambda(t)h(t - F_{i-1})dt \right)
\]

(8)

Comparing equation (4) of the main text, we have the desired result.