We first prove equation (4) of the main text for a general nonstationary hazard function $h(\tau, t)$.

**Proposition S.1** For a renewal process with nonstationary hazard function $h(\tau, t)$, the waiting time $\tau$ given that the last event occurred at time $t_{\text{prev}}$ is given by

$$g(\tau | t_{\text{prev}}) = h(\tau, t_{\text{prev}} + \tau) \exp \left( - \int_0^\tau h(u, t_{\text{prev}} + u) du \right)$$  \hspace{1cm} (1)

**Proof.** By definition (see equation (2) in the main text),

$$h(\tau, t_{\text{prev}} + \tau) = \frac{g(\tau | t_{\text{prev}})}{1 - \int_0^\tau g(u | t_{\text{prev}}) du}$$  \hspace{1cm} (2)

Let $y = 1 - \int_0^\tau g(u | t_{\text{prev}}) du$. It follows that

$$h(\tau, t_{\text{prev}} + \tau) = \frac{-dy/d\tau}{y}, \text{ so that}$$

$$y = \exp \left( - \int_0^\tau h(u, t_{\text{prev}} + u) du \right)$$  \hspace{1cm} (4)

Substituting back for $y$ and differentiating w.r.t. $\tau$, we get equation (??).

We now prove proposition 2 from the main text.

**Proposition 2** For any $\Omega \geq \max_{1, \tau} h(\tau) \lambda(t)$, $F$ is a sample from a modulated renewal process with hazard $h(\cdot)$ and modulating intensity $\lambda(\cdot)$.

**Proof.** We need to show that $F_i - F_{i-1} \sim g$.

Denote by $E_i^*$ the restriction of $E$ to the interval $(F_{i-1}, F_i)$, not including boundaries. Note that

$$P(F_i, E_i^* | F_{i-1}) = \left( \prod_{e \in E_i^*} 1 - \frac{\lambda(e) h(e - F_{i-1})}{\Omega} \right) \frac{\lambda(F_i) h(F_i - F_{i-1})}{\Omega}$$  \hspace{1cm} (5)
Defining $n = |E^*_i|$ and $t_0 = F_{i-1}$, we have

\[
P(F_i, n|F_{i-1}) = \frac{\lambda(F_i)h(F_i - F_{i-1})}{\Omega} \int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} \int_{t_{n-1}}^{F_i} dt_1 dt_2 ... dt_n \left( \prod_{j=1}^{n} \Omega \exp(-\Omega(t_j - t_{j-1})) \right) \left( \prod_{j=1}^{n} \frac{1 - \frac{\lambda(t_j)h(t_j - F_{i-1})}{\Omega}}{\Omega} \right) (\Omega \exp(-\Omega(F_i - t_n)))
\]

\[
= \lambda(F_i)h(F_i - F_{i-1}) \exp(-\Omega(F_i - F_{i-1})) \int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} \int_{t_{n-1}}^{F_i} dt_1 dt_2 ... dt_n \left( \prod_{j=1}^{n} (\Omega - \lambda(t_j)h(t_j - F_{i-1})) \right)
\]

Marginalizing out $n$, we then have

\[
P(F_i|F_{i-1}) = \lambda(t)h(F_i - F_{i-1}) \exp(-\Omega(F_i - F_{i-1})) \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{F_{i-1}}^{F_i} dt (\Omega - \lambda(t)h(t - F_{i-1})) \right)^n \right)
\]

\[
= \lambda(F_i)h(F_i - F_{i-1}) \exp \left(- \int_{F_{i-1}}^{F_i} \lambda(t)h(t - F_{i-1}) dt \right)
\]

Comparing equation (4) of the main text, we have the desired result.

\[\square\]