Information Theory

Maneesh Sahani

Gatsby Computational Neuroscience Unit
University College London

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Quantifying a Code

- How much information does a neural response carry about a stimulus?
- How efficient is a hypothetical code, given the statistical behaviour of the components?
- How much better could another code do, given the same components?
- Is the information carried by different neurons complementary, synergistic (whole is greater than sum of parts), or redundant?
- Can further processing extract more information about a stimulus?

Information theory is the mathematical framework within which questions such as these can be framed and answered.

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Information theory does not directly address:

- estimation (but there are some relevant bounds)
- computation (but “information bottleneck” might provide a motivating framework)
- representation (but redundancy reduction has obvious information theoretic connections)

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We need to start by considering the uncertainty in a probability distribution $\to$ called the **entropy**

Let $S \sim P(S)$. The entropy is the minimum number of bits needed, on average, to specify the value S takes, assuming $P(S)$ is known.

Equivalently, the minimum average number of yes/no questions needed to guess S.

Entropy

Entropy

- Suppose there are M equiprobable stimuli: $P(s_m) = 1/M$.

  To specify which stimulus appears on a given trial, we would need assign each a (binary) number. This would take,

$$
\begin{aligned}
B_s &\leq \quad \log_2 M + 1 \qquad [2^B \geq M] \\
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- Now suppose we code N such stimuli, drawn iid, at once.

$$\log_2 M^N \quad B_N \leq \log_2 M^N + 1 \qquad 2^{B_N} \geqslant M^N$$
$$\rightarrow -N \log_2 \frac{1}{M} \qquad \text{as } N \rightarrow \infty$$
$$\Rightarrow B_s \rightarrow -\log_2 p \text{ bits}$$

  This is called block coding. It is useful for extracting theoretical limits. The nervous system is unlikely to use block codes in time, but may in space.

Entropy

- Now suppose stimuli are not equiprobable. Write $P(s_m) = p_m$. Then

$$P(S_1, S_2, \ldots, S_N) = \prod_m p_m^{n_m} \qquad\qquad [\text{where } n_m = (\text{\# of } S_i = s_m)].$$

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Rather than appealing to typicality, we could instead have used the law of large numbers directly:

$$\frac{1}{N} \log_2 P(S_1, S_2, \ldots S_N) = \frac{1}{N} \log_2 \prod_i P(S_i) = \frac{1}{N} \sum_i \log_2 P(S_i) \overset{N \to \infty}{\to} \mathrm{E}[\log_2 P(S_i)]$$

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$$P(S|r) = \frac{P(r|S)P(S)}{\sum_s P(r|s)P(s)}$$

so we can write

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The *average* uncertainty in S for $r \sim P(R) = \sum_s P(R|s)p(s)$ is then

$$\mathbf{H}[S|R] = \sum_r P(r) \left[-\sum_s P(s|r) \log_2 P(s|r) \right] = -\sum_{s,r} P(s,r) \log_2 P(s|r)$$

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It is easy to show that:

$$-\sum_{s,r} P(s,r) \log_2 P(s,r)$$

1. $\mathbf{H}[S|R] \leq \mathbf{H}[S]$
2. $\mathbf{H}[S|R] = \mathbf{H}[S,R] - \mathbf{H}[R]$
3. $\mathbf{H}[S|R] = \mathbf{H}[S]$ iff $S \perp\!\!\!\perp R$

Average Mutual Information

A natural definition of the average information gained about S from R is

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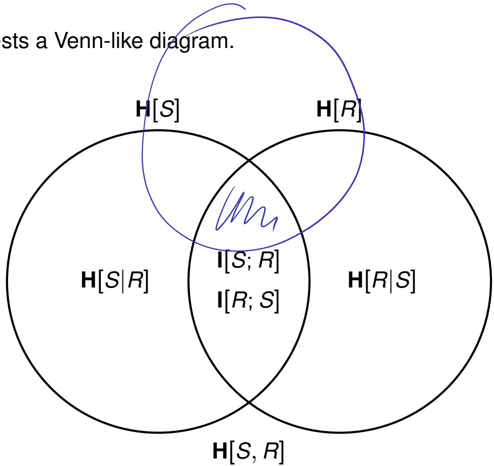
It follows from the definition that

$$\begin{aligned}
\mathbf{I}[S;R] &= \sum_{s} P(s) \log \frac{1}{P(s)} - \sum_{s,r} P(s,r) \log \frac{1}{P(s|r)} \\
&= \sum_{s,r} P(s,r) \log \frac{1}{P(s)} + \sum_{s,r} P(s,r) \log P(s|r) \\
&= \sum_{s,r} P(s,r) \log \frac{P(s|r)}{P(s)} \\
&= \sum_{s,r} P(s,r) \log \frac{P(s,r)}{P(s)P(r)} \\
&= \mathbf{I}[R;S]
\end{aligned}$$

Average Mutual Information

The symmetry suggests a Venn-like diagram.



All of the additive and equality relationships implied by this picture hold for two variables. Unfortunately, we will see that this does not generalise to any more than two.

Kullback-Leibler Divergence

Another useful information theoretic quantity measures the difference between two distributions.

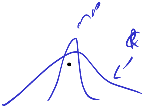
$$H[P] = \underbrace{\sum_s P(s)}_{\mathbb{E}} \underbrace{\log \frac{1}{P(s)}}_{\text{codeword length}}$$

$$\begin{aligned}
\mathbf{KL}[P(S) \| Q(S)] &= \sum_s P(s) \log \frac{P(s)}{Q(s)} \\
&= \underbrace{\sum_s P(s) \log \frac{1}{Q(s)}}_{\text{cross entropy}} - \mathbf{H}[P]
\end{aligned}$$

$\mathbb{E}$

codeword length under Q

Excess cost in bits paid by encoding according to Q instead of P.



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Excess cost in bits paid by encoding according to Q instead of P.

$$\begin{aligned}
-\mathbf{KL}[P\|Q] &= \sum_s P(s) \log \frac{Q(s)}{P(s)} \\
&\leq \log \sum_s P(s) \frac{Q(s)}{P(s)} \qquad \text{by Jensen} \\
&= \log \sum_s Q(s) = \log 1 = 0
\end{aligned}$$

So $\mathbf{KL}[P\|Q] \geq 0$. Equality iff $P = Q$

Mutual Information and KL

$$\mathbf{I}[S;R] = \sum_{s,r} P(s,r) \log \frac{P(s,r)}{P(s)P(r)} = \mathbf{KL}[P(S,R) \| P(S)P(R)]$$

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Thus:

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Thus:

1. Mutual information is always non-negative

$$\mathbf{I}[S;R] \geq 0$$

2. Conditioning never increases entropy

$$\mathbf{H}[S|R] \leq \mathbf{H}[S]$$

Multiple Responses

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$$I_{12} = \mathbf{I}[S; R_1, R_2] = \mathbf{H}[R_1, R_2] - \mathbf{H}[R_1, R_2|S]$$

$$R_1 \perp\!\!\!\perp R_2 \Rightarrow \mathbf{H}[R_1, R_2] = \mathbf{H}[R_1] + \mathbf{H}[R_2]$$

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S	R_1	R_2	
0	0	1	$P(0,1\|0) = 0.5$
	1	0	$P(1,0\|0) = 0.5$
1	0	0	
	1	1	

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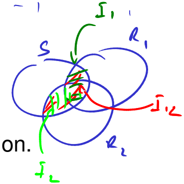

$I_{12} > \max(I_1, I_2)$: the second response cannot destroy information.

Thus, the Venn-like diagram with three variables is misleading.

Data Processing Inequality

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Previous case : $S \to R_1$, $S \to R_2$

Suppose $S \to R_1 \to R_2$ form a Markov chain; that is, $R_2 \perp\!\!\!\perp S \mid R_1$.

Then,

$$
\begin{aligned}
& P(R_2, S|R_1) = P(R_2|R_1)P(S|R_1) \\
\Rightarrow\ & P(S|R_1, R_2) = P(S|R_1)
\end{aligned}
$$

Data Processing Inequality

$$\begin{tikzcd} S \arrow[r] & R_1 \arrow[r] \arrow[d] & R_2 \arrow[d] \\ & D_1 & D_2 \end{tikzcd}$$

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Thus,

$$\mathbf{H}[S|R_2] \geq \mathbf{H}[S|R_1, R_2] = \mathbf{H}[S|R_1]$$

$$\Rightarrow \mathbf{I}[S; R_2] \leq \mathbf{I}[S; R_1]$$

So any computation based on R_1 that does not have separate access to S cannot add information (in the Shannon sense) about the world.

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So any computation based on R_1 that does not have separate access to S cannot add information (in the Shannon sense) about the world.

Equality holds iff $S \to R_2 \to R_1$ as well. In this case R_2 is called a **sufficient statistic** for S.

Entropy Rate

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Let $\mathcal{S} = \{S_1, S_2, S_3 \ldots\}$ form a stochastic process.

$$\begin{aligned}\mathbf{H}[S_1, S_2, \ldots, S_n] &= \mathbf{H}[S_n | S_1, S_2, \ldots, S_{n-1}] + \mathbf{H}[S_1, S_2, \ldots, S_{n-1}] \\ &= \mathbf{H}[S_n | S_1, S_2, \ldots, S_{n-1}] + \mathbf{H}[S_{n-1} | S_1, S_2, \ldots, S_{n-2}] + \ldots + \mathbf{H}[S_1]\end{aligned}$$

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The **entropy rate** of $\mathcal{S}$ is defined as

$$\mathbf{H}[\mathcal{S}] = \lim_{n\to\infty} \frac{\mathbf{H}[S_1, S_2, \ldots, S_n]}{n}$$

or alternatively as

$$\mathbf{H}[\mathcal{S}] = \lim_{n\to\infty} \mathbf{H}[S_n|S_1, S_2, \ldots, S_{n-1}]$$

Entropy Rate

So far we have discussed S and R as single (or iid) random variables. But real stimuli and responses form a time series.

Let $\mathcal{S} = \{S_1, S_2, S_3 \ldots\}$ form a stochastic process.

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or alternatively as

$\lim_{n\to\infty} I(S_n ; R_n | S_1 .. S_{n-1}, R_1 .. R_{n-1})$

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If $S_i \stackrel{\text{iid}}{\sim} P(S)$ then $\mathbf{H}[\mathcal{S}] = \mathbf{H}[S]$.
If $\mathcal{S}$ is Markov (and stationary) then $\mathbf{H}[\mathcal{S}] = \mathbf{H}[S_n|S_{n-1}]$.

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The discussion so far has involved discrete S and R. Now, let $S \in \mathbb{R}$ with density p(s). What is its entropy?

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We define the **differential entropy**:

$$h(S) = -\int ds \; p(s) \log p(s).$$

Note that $h(S)$ can be < 0, and can be $\pm\infty$.

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The conditional differential entropy is

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and, like the differential entropy itself, may be poorly behaved.

The mutual information, however, is well-defined

$$\begin{aligned}
\mathrm{I}_\Delta[S;R] &= \mathrm{H}_\Delta[S] - \mathrm{H}_\Delta[S|R] \\
&= -\sum_i \Delta s\; p(s_i) \log p(s_i) - \log \Delta s \\
&\qquad - \int dr\; p(r) \left(-\sum_i \Delta s\; p(s_i|r) \log p(s_i|r) - \log \Delta s \right) \\
&\to h(S) - h(S|R)
\end{aligned}$$

$$= \int dr\, ds\; p(r,s) \log \frac{p(r,s)}{p(r)\,p(s)}$$

as are other KL divergences.

Maximum Entropy Distributions

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\mathcal{L} &= \int ds\; p(s) \log p(s) - \lambda_0 \left[\int ds\; p(s) - 1 \right] - \lambda_1 \left[\int ds\; p(s) f(s) - a \right] \\
\frac{\delta \mathcal{L}}{\delta p(s)} &= 1 + \log p(s) - \lambda_0 - \lambda_1 f(s) = 0 \\
\Rightarrow \log p(s) &= \lambda_0 + \lambda_1 f(s) - 1 \\
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The constants λ_0 and λ_1 can be found by solving the constraint equations.

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$$\begin{array}{lcll} f(s) = s & \Rightarrow & p(s) = \frac{1}{Z} e^{\lambda_1 s}. & \text{Exponential (need } p(s) = 0 \text{ for } s < T\text{).} \\ f(s) = s^2 & \Rightarrow & p(s) = \frac{1}{Z} e^{\lambda_1 s^2}. & \text{Gaussian.} \end{array}$$

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Both results together $\Rightarrow$ maximum entropy point process (for fixed mean arrival rate) is homogeneous Poisson – independent, exponentially distributed ISIs.

Channels

We now direct our focus to the conditional $P(R|S)$ which defines the **channel** linking S to R.

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The mutual information

$$\mathbf{I}[S;R] = \sum_{s,r} P(s,r) \log \frac{P(s,r)}{P(s)P(r)} = \sum_{s,r} {\color{red}P(s)} P(r|s) \log \frac{P(r|s)}{{\color{red}P(r)}}$$

depends on marginals $P(s)$ and $P(r) = \sum_s P(r|s)P(s)$ as well and thus is unsuitable to characterise the conditional alone.

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Instead, we characterise the channel by its **capacity**

$$\mathbf{C}_{R|S} = \sup_{P(s)} \mathbf{I}[S; R]$$

Thus the capacity gives the theoretical limit on the amount of information that can be transmitted over a channel. Clearly, this is limited by the properties of the noise.

Joint source-channel coding theorem

The remarkable central result of information theory.

$$S \xrightarrow{\text{encoder}} \widetilde{S} \xrightarrow[\mathbf{C}_{R|\widetilde{S}}]{\text{channel}} R \xrightarrow{\text{decoder}} \widehat{T}$$

Any source ensemble S with entropy $\mathbf{H}[S] < \mathbf{C}_{R|\widetilde{S}}$ can be transmitted (in sufficiently long blocks) with $P_{error} \to 0$.

The proof is beyond our scope.

Some of the key ideas that appear in the proof are:

- block coding
- error correction
- joint typicality
- random codes

The channel coding problem

$$S \xrightarrow{\textit{encoder}} \widetilde{S} \xrightarrow[\mathbf{C}_{R|\widetilde{S}}]{\textit{channel}} R \xrightarrow{\textit{decoder}} \widehat{T}$$

Given channel $P(R|\widetilde{S})$ and source $P(S)$, find **encoding** $P(\widetilde{S}|S)$ (may be deterministic) to maximise $\mathbf{I}[S; R]$.

By data processing inequality, and defn of capacity:

$$\mathbf{I}[S; R] \leq \mathbf{I}[\widetilde{S}; R] \leq \mathbf{C}_{R|\widetilde{S}}.$$

By JSCT, equality can be achieved (in the limit of increasing block size).

Thus $\mathbf{I}[\widetilde{S}; R]$ should saturate $\mathbf{C}_{R|\widetilde{S}}$.

See homework for an algorithm (Blahut-Arimoto) to find $P(\widetilde{S})$ that saturates $\mathbf{C}_{R|\widetilde{S}}$ for a general discrete channel.

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i.e.

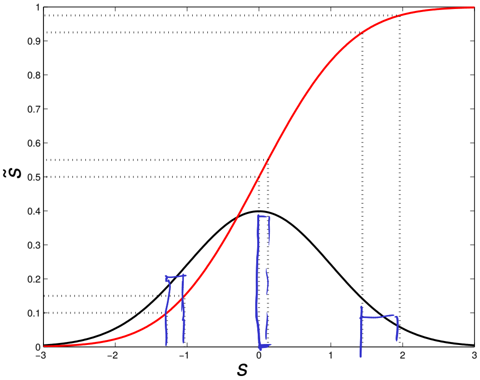
$$p(r) = \begin{cases} \frac{1}{r_{\max}} & r \in [0, r_{\max}] \\ 0 & \text{otherwise} \end{cases}$$

Histogram Equalisation

Suppose $r = \tilde{s} + \eta$ where η represents a (relatively small) source of noise. Consider deterministic encoding $\tilde{s} = f(s)$. How do we ensure that $p(r) = 1/r_{\max}$?

$$\frac{1}{r_{\max}} = p(r) \approx p(\tilde{s}) = \frac{p(s)}{f'(s)} \qquad \Rightarrow f'(s) = r_{\max}\; p(s)$$

$$\Rightarrow f(s) = r_{\max} \int_{-\infty}^{s} ds'\; p(s')$$



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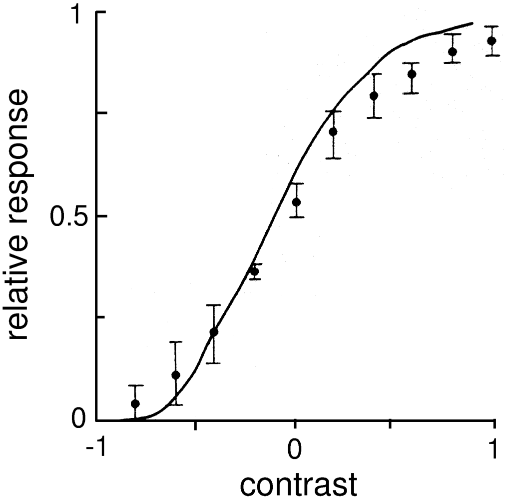

Laughlin (1981)

Gaussian channel

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then,

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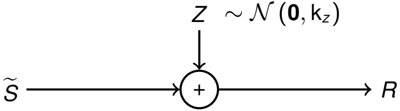
Gaussian channel – white noise



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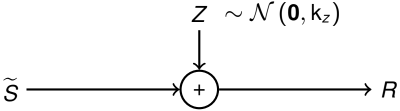

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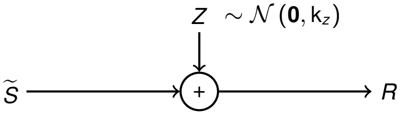

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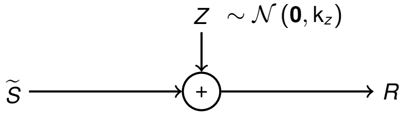

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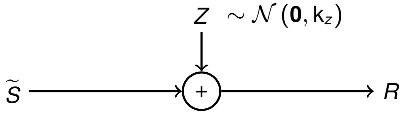

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\mathbf{I}[\widetilde{S}; R] &= h(R) - h(R|\widetilde{S}) \\
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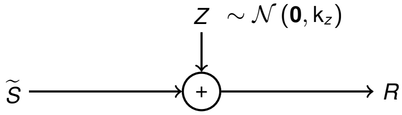

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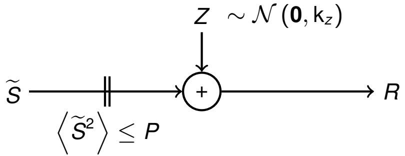

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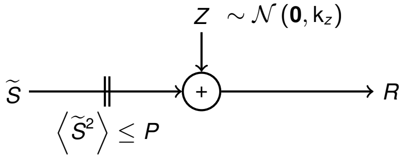

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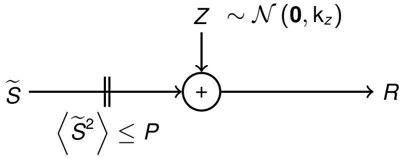

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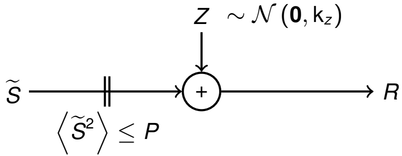

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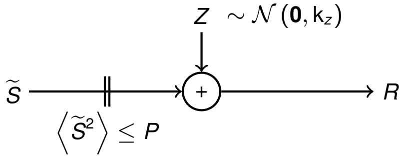

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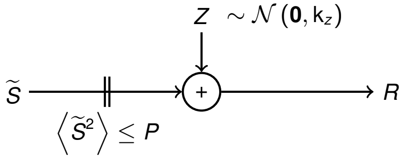

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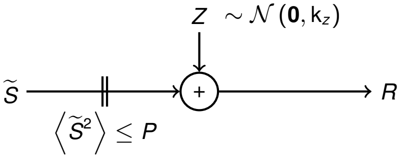

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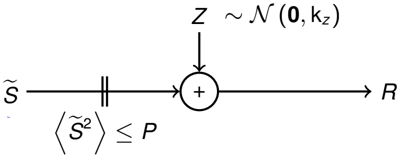
$$

Gaussian channel – white noise



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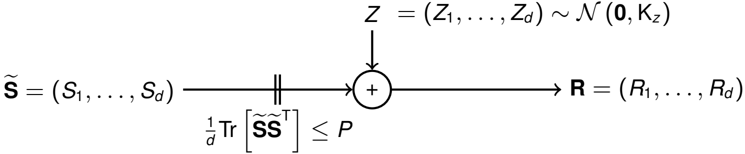
\end{aligned}
$$

$$
\mathbf{C}_{R|\widetilde{S}} = \frac{1}{2} \log 2\pi e \left(1 + \frac{\mathrm{P}}{\mathrm{k}_z}\right)
$$

The capacity is achieved iff $R \sim \mathcal{N}(0, \mathrm{P} + \mathrm{k}_z) \quad \Rightarrow \widetilde{S} \sim \mathcal{N}(0, \mathrm{P})$.

Gaussian channel – correlated noise

Now consider a vector Gaussian channel:



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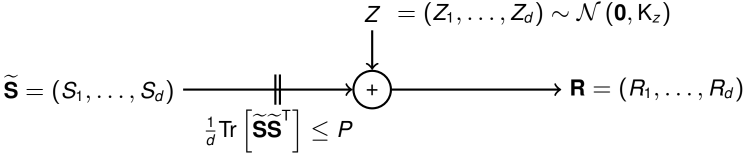

Following the same approach as before:

$$\mathbf{I}[\widetilde{\mathbf{S}}; \mathbf{R}] = h(\mathbf{R}) - h(\mathbf{Z}) \leq \frac{1}{2}\log\left[(2\pi e)^d \ |\mathrm{K}_{\tilde{s}} + \mathrm{K}_z|\right] - \frac{1}{2}\log\left[(2\pi e)^d \ |\mathrm{K}_z|\right],$$

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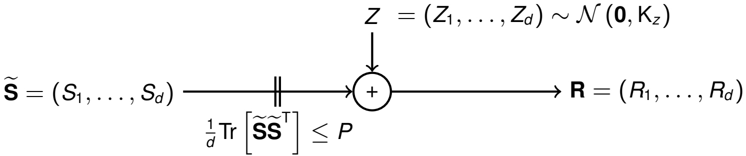

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$\Rightarrow \mathbf{C}_{R|S}$ achieved when $\widetilde{\mathbf{S}}$ (and thus $\mathbf{R}$) $\sim \mathcal{N}$, with $|\mathrm{K}_{\tilde{s}} + \mathrm{K}_z|$ max given $\frac{1}{d}\mathrm{Tr}\left[\mathrm{K}_{\tilde{s}}\right] \leq P$.

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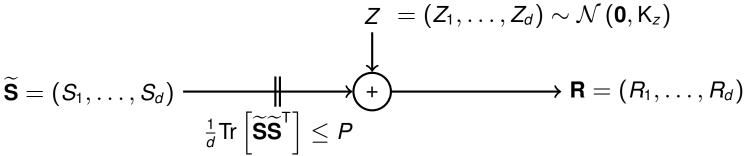

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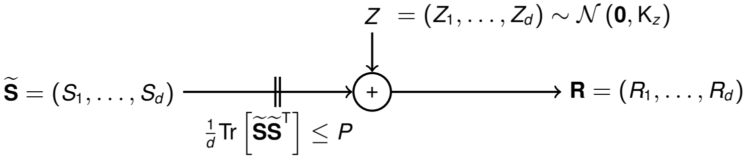

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For **stationary** noise (wrt dimension indexed by d) this can be achieved by a Fourier transform $\Rightarrow$ index diagonal elements by ω.

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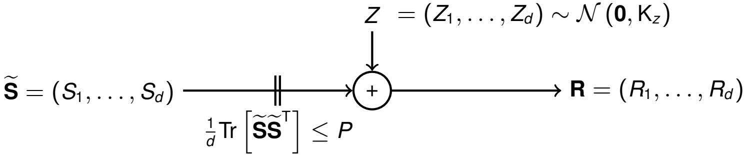

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$$\mathsf{k}^*_{\tilde{s}}(\omega) = \mathrm{argmax} \prod_{\omega} \left(\mathsf{k}_{\tilde{s}}(\omega) + \mathsf{k}_z(\omega)\right) \qquad \text{such that } \frac{1}{d}\sum \mathsf{k}_{\tilde{s}}(\omega) \leq P$$

Water filling

Assume that optimum is achieved for max. input power.

$$\mathrm{k}_{\tilde{s}}^{*}(\omega) = \operatorname{argmax} \left[\sum_{\omega} \log \left(\mathrm{k}_{\tilde{s}}(\omega) + \mathrm{k}_{z}(\omega) \right) - \lambda \left(\frac{1}{d} \sum_{\omega} \mathrm{k}_{\tilde{s}}(\omega) - P \right) \right]$$

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(\mathrm{k}_{\tilde{s}}\geq 0) &\Rightarrow\ \mathrm{k}_{\tilde{s}}^{*}(\omega)=\left[\nu-\mathrm{k}_{z}(\omega)\right]^{+}
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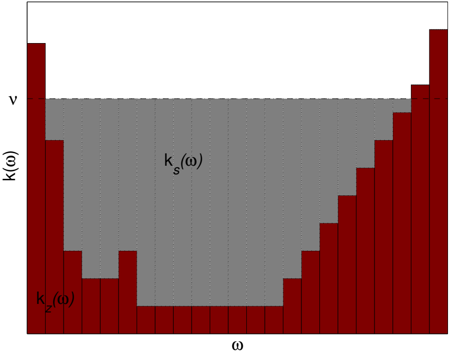
$$(\mathrm{k}_{\tilde{s}} \geq 0) \Rightarrow \quad \mathrm{k}^*_{\tilde{s}}(\omega) = [\nu - \mathrm{k}_z(\omega)]^+$$

Waterfilling: choose ν so

$$\sum_\omega \mathrm{k}_{\tilde{s}}(\omega) = d \cdot P$$



R is white or decorrelated (within power budget) $\Rightarrow$variance equalisation.

Decorrelation at the retina

$s \longrightarrow \tilde{s} \xrightarrow{c} r$

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Therefore the correlation (covariance) between cells is

$= \langle (r_a - \langle r_a \rangle)(r_b - \langle r_b \rangle) \rangle$

$$\begin{aligned} Q_r(\mathbf{a},\mathbf{b}) &= \left\langle \int d\mathbf{x}\ d\mathbf{y}\ D_s(\mathbf{x}-\mathbf{a}) D_s(\mathbf{y}-\mathbf{b}) s(\mathbf{x}) s(\mathbf{y}) \right\rangle \\ &= \int d\mathbf{x}\ d\mathbf{y}\ D_s(\mathbf{x}-\mathbf{a}) D_s(\mathbf{y}-\mathbf{b})\ \underbrace{\langle s(\mathbf{x}) s(\mathbf{y}) \rangle}_{Q_s(\mathbf{x},\mathbf{y})} \end{aligned}$$

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Using (spatial) stationarity, we can transform to the Fourier domain:

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and thus output decorrelation requires

$$|\widetilde{D}_s(\mathbf{k})|^2 \propto \frac{1}{\widetilde{Q}_s(\mathbf{k})}$$

Decorrelation at the retina

$\frac{1}{f}$ $\frac{1}{f^2}$

Spatial correlations of natural images fall off with f^{-2}:

$$\widetilde{Q}_s(\mathbf{k}) \propto \frac{1}{|\mathbf{k}|^2 + k_0^2}$$

and the optical filter of the eye introduces (crudely) a low-pass term $\propto e^{-\alpha|\mathbf{k}|}$.
So decorrelation requires

$$|\widetilde{D}_s(\mathbf{k})|^2 \propto \frac{|\mathbf{k}|^2 + k_0^2}{e^{-\alpha|\mathbf{k}|}}$$

$\tilde{Q}_s$

optics

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Photodetection introduces noise. Therefore, cascade linear filters:

$$\mathbf{s} + \boldsymbol{\eta} \xrightarrow[D_\eta]{} \hat{\mathbf{s}} \xrightarrow[D_s]{} \mathbf{r}$$

$s, \eta \to s$

with

$$\widetilde{D}_\eta(\mathbf{k}) = \frac{\widetilde{Q}_s(\mathbf{k})}{\widetilde{Q}_s(\mathbf{k}) + \widetilde{Q}_\eta(\mathbf{k})} \qquad \text{(Wiener filter)}$$

$$\frac{\mathrm{cov}(x, y)}{\mathrm{cov}(x)}$$

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So decorrelation requires

$$|\widetilde{D}_s(\mathbf{k})|^2 \propto \frac{|\mathbf{k}|^2 + k_0^2}{e^{-\alpha|\mathbf{k}|}} = \frac{1}{\tilde{Q}_s(k)\, e^{-\alpha|k|}}$$

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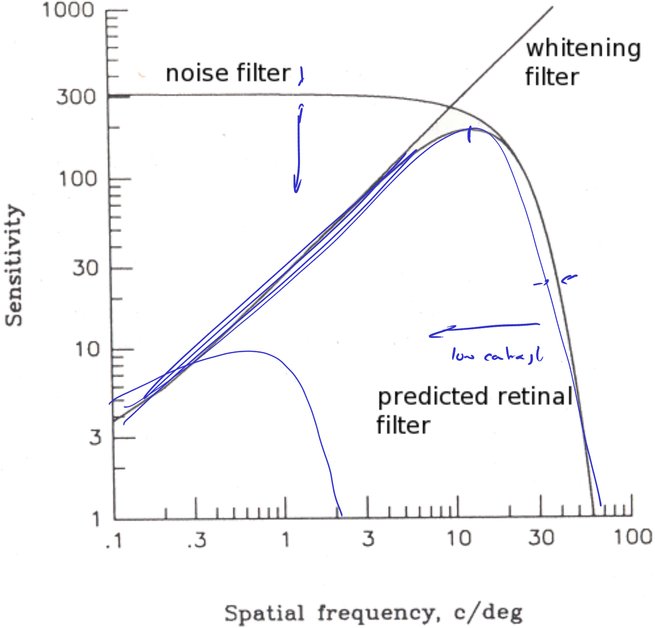
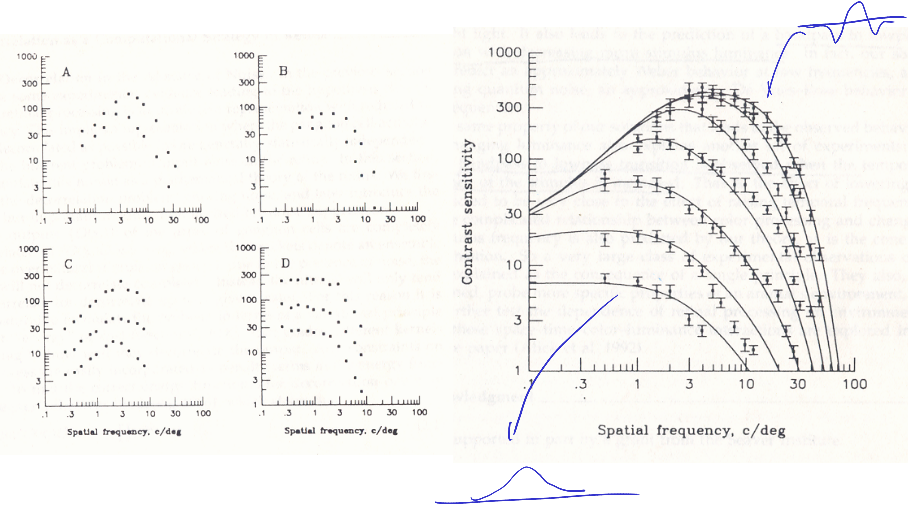
Thus the combined RGC filter is predicted to be:

$$|\widetilde{D}_s(\mathbf{k})|\widetilde{D}_\eta(\mathbf{k}) \propto \frac{\sqrt{\widetilde{Q}_s(\mathbf{k})}}{\widetilde{Q}_s(\mathbf{k}) + \widetilde{Q}_\eta(\mathbf{k})}$$

Decorrelation at the retina



Decorrelation at the retina



Related ideas

- efficient channel utilisation
- output entropy maximisation
- variance equalisation
- redundancy reduction
- decorrelation
- discovery of independent projections or components